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Fundamental Constants in Singularity-Free Five-Dimensional Kaluza–Klein Cosmological Model

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Expressions for the time dependence of the fundamental constants are derived through dimensional reduction and one-loop quantum corrections to scalar fields. Moreover, singularity-free solutions of Einstein's field equations are obtained. Using these solutions, we discuss the time dependence of fundamental constants. It is interesting to see that the fine structure constant asymptotically approaches to $1/137$, G_{eff} (effective four-dimensional constant) approaches G_{N} (Newtonian gravitational constant), and Λ_{eff} vanishes. Graphical representations of these results are also given for a special case.

1. INTRODUCTION

In the context of the unification of gravity with other forces of nature, Kaluza–Klein theory (Kaluza, 1921; Klein, 1926*a*,*b*; Duff *et al.*, 1986) is important. Though much work has been done on this theory, some results in the effective four-dimensional cosmological model (obtained through compactification of extra dimensions) are interesting. In this theory, the five-dimensional manifold is a product of M^4 (usual four-dimensional paracompact manifold) and a compact manifold S^1 (circle). Our observable universe is four-dimensional, so at late times the radius of the extra manifold (S^1) is supposed to be extremely small (undetectable). Hence, it is very natural to think that if the extra manifold really existed at very high energy scale in the extreme past and is undetectable at late times due to the nonavailability of the energy required to detect it, it should manifest itself in some way or other. Employing the heat-kernel method, Toms (1983) calculated the one-loop effective action in a five-dimensional background geometry and obtained the induced cosmological constant, gravity, and

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Maxwell's term as manifestation of the fifth dimension of the space. The model considered by Toms has a static component of the metric tensor corresponding to the extra dimension. Later, Copeland and Toms (1985) calculated the time-dependent gravitational as well as cosmological term through the one-loop quantum correction to the dimensionally reduced action for scalar fields on a higher-dimensional time-dependent Kaluza–Klein background. Gleiser and Taylor (1983) also discussed the time dependence of the gravitational as well as cosmological constant in a six dimensional cosmological background without including the contribution of the one-loop correction of scalar or spinor fields. In the context of Kaluza–Klein theories, Marciano (1984) also discussed the time dependence of fundamental constants and suggested that the time variation of these constants may provide evidence for the existence of the internal manifold. Gilbert *et al.* (1984) calculated the one-loop effective action for scalar fields, and Gilbert and McClain (1984) calculated the same for spinors, in time-dependent Kaluza–Klein background.

As discussed in Section 4 of this paper, fundamental constants (gravitational constant, cosmological constant, and fine structure constant) greatly depend on the scale factor $b(t)$ associated with the internal manifold. So, $b(t)$ is very crucial to getting natural values (observed values) of these constants today. Naturally $b(t)$ is a solution of higher-dimensional Einstein field equations. Most of the solutions obtained (corresponding to $b(t)$) suffer from the disease of "crack of doom" singularity. For example, in the Chodos–Detweiler (1986) model it occurs at $t = \infty$. Matzner and Mezzacappa (1986) investigated a five-dimensional model in which this problem occurs at some time during the evolution of the model. Solutions obtained by Sahadev (1984) also suffer from this disease. As realized in Section 4, $b(t)$ should asymptotically stabilize. Copeland and Toms (1984) also realized the necessity of this constraint, but they found that the solution is unstable near the asymptotic limit of $b(t)$. Actually, Copeland and Toms did not solve the $(4 + d)$-dimensional Einstein equations, but solved equations derived from the effective four-dimensional action for gravity. Gleiser and Taylor (1983) solved six-dimensional Einstein equations (assuming some suitable equations of state for matter) such that their solutions satisfy the above criterion.

Rosenbaum *et al.* (1987) suggested diagrammatic solutions for a five-dimensional Kaluza–Klein cosmological model and showed that by an appropriate choice of parameters the "crack of doom" singularity can be avoided. Here, solutions of Einstein's equations are obtained which are free from the "crack of doom" singularity and the external manifold is also singularity-free, i.e., the scale factor associated with the three-dimensional subspace of the external manifold is initially nonvanishing and the scale-factor (associated with the internal manifold) asymptotically stabilizes at a nonzero constant value. As a result time-dependent fundamental constants approach the required numerical values as time increases.

This paper is organized as follows. Section 2 contains the five-dimensional model and the action used in the theory. Section 3 discusses the dimensional reduction of fields. In Section 4, a one-loop correction for the scalar field is discussed and expressions showing a time dependence of fundamental constants are derived. In Section 5, the five-dimensional Einstein field equations are solved. Section 6 discusses results in the light of the solution of Einstein's equations obtained in the preceding section.

Natural units $\hbar = c = 1$ are used throughout the paper, where $\hbar$ and c have their usual meaning. An overdot denotes $\partial/\partial t$ (t denotes time).

2. MODEL

We consider a five-dimensional space with coordinates x^μ ($\mu = 0, 1, 2, 3$) of the ordinary four-dimensional manifold M^4 and coordinates y on the internal manifold (circle). The line element is written as

$$ds^2 = dt^2 - \frac{a^2(t)}{(1+\epsilon r^2/4)}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] - b^2(t)[dy - kA_\mu(x)\,dx^\mu]^2 \tag{2.1}$$

where t is the cosmic time, $a(t)$ is the scale factor for the $t = \text{const}$ hypersurface of M^4, ϵ is the curvature constant (with possible values $+1$, 0, and -1 corresponding to closed, flat and open models of $t = \text{const}$ hypersurface of M^4), $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$, $A_\mu(x)$ is the four-dimensional electromagnetic field, and k is a constant of dimension $(\text{mass})^{-1}$ to make $kA_\mu(x)$ dimensionless.

In the horizontal lift basis (HLB) the metric tensor (Misner *et al.*, 1973; Huggins and Toms, 1986) given by (2.1) is written as

$$g_{m'n'} = \text{diag}\left(1, -\frac{a^2(t)}{(1+\epsilon r^2/4)^2}, -\frac{a^2(t)}{(1+\epsilon r^2/4)^2}, -\frac{a^2(t)}{(1+\epsilon r^2/4)^2}, -b^2(t)\right) \tag{2.2}$$

Using the HLB, we write the total action of the theory as

$$S = \frac{1}{16\pi G_5}\int d^4x\,dy\,(-g_5)^{1/2}R_5 + \frac{1}{2}\int d^4x\,dy\,(-g_5) \times [g^{m'n'}(D_{m'}\phi)^*(D_{n'}\phi) - \mathscr{E}R_5\phi^*\phi] + S^{(m)}$$

where $G_5 = \bar{G}L$, ($\bar{G}$ is proportional to G_N, the Newtonian gravitational constant, which is equal to M_P^{-2}, where M_P is the Planck mass, $0 \le y \le L$),

R_5 is a five-dimensional Ricci scalar, $R_5 = R_4 - \frac{1}{4}k^2 F_{\mu\nu}F^{\mu\nu}$ (R_4 is a four-dimensional Ricci scalar, $F_{\mu\nu} = D_\nu A_\mu - D_\mu A_\nu$, $D_\mu = \nabla_\mu + kA_\mu$, and ∇_μ and ∇_5 are covariant derivatives in curved space), $\mathscr{E}$ is the coupling constant, ϕ is the scalar field, $S^{(m)}$ is the action for the matter (perfect fluid) other than ϕ, and g_5 is the determinant of the metric tensor $g_{m'n'}$ given by (2.2).

3. DIMENSIONAL REDUCTION

3.1. Gravity

The five-dimensional action for gravity as given in the total action (2.3) is

$$S_g^{(5)} = -\frac{1}{16\pi G_5}\int d^4x\, dy\, (-g_5)^{1/2}\left(R_4 - \frac{1}{4}k^2 F_{\mu\nu}F^{\mu\nu}\right) \tag{3.1}$$

For the sake of convenience, the metric given by (2.2) is written as

$$g_{MN} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & -b^2(t) \end{pmatrix} \tag{3.2}$$

where

$$g_{\mu\nu} \equiv \operatorname{diag}\left(1, -\frac{a^2(t)}{(1+\epsilon r^2/4)^2}, -\frac{a^2(t)}{(1+\epsilon r^2/4)^2}, -\frac{a^2(t)}{(1+\epsilon r^2/4)^2}\right)$$

We conformally transform g_{MN} to $\tilde{g}_{MN}$ as

$$g_{MN} = b^2(t)\tilde{g}_{MN} = b^2(t)\begin{pmatrix} \tilde{g}_{\mu\nu} & 0 \\ 0 & -1 \end{pmatrix} \tag{3.3}$$

where $\tilde{g}_{\mu\nu} = b^2(t) g_{\mu\nu}$. So, ignoring terms of total divergence and integrating over y, one gets

$$S_g^{(4)} = -\frac{1}{16\pi \tilde{G}}\int d^4x\, (-\tilde{g}_4)^{1/2} b^3\left[\tilde{R}_4 - \frac{k^2 b^{-2}}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} - 12\left(\frac{\dot{b}}{b}\right)^2\right] \tag{3.4}$$

Further, to undo the earlier conformal transformation on $g_{\mu\nu}$, another conformal transformation is done as

$$\tilde{g}_{\mu\nu} = \bar{b}^2(t) g_{\mu\nu} \tag{3.5}$$

Under (3.5), the action (3.4) is written as

$$S_g^{(4)} = -\frac{1}{16\pi \tilde{G}}\int d^4x\, (-g_4)^{1/2} b\left[R_4 - \frac{k^2}{4}F_{\mu\nu}F^{\mu\nu}\right] \tag{3.6}$$

3.2. Scalar Fields

The internal manifold is a circle, which is not simply connected, hence any field on it can be either untwisted (periodic in y) or twisted (antiperiodic in y). Hence, in either case, one may write (Toms, 1983)

$$\phi(x^\mu, y) = [Lb(t)]^{1/2} \sum_{n=-\infty}^{\infty} \phi_n(x^4)\exp[i(n+\alpha)My] \tag{3.7}$$

where $M = 2\pi/L$ (L is the circumference of the circle, the internal manifold) and $\alpha = 0$ (1/2) for the untwisted (twisted) field.

Substituting ϕ given by (3.7) in the action for the scalar field in (2.3) and integrating over y, we obtain

$$S_\phi^{(4)} = \frac{1}{2}\sum_{n=-\infty}^{\infty}\int d^4x\,[-g_4(x)]^{1/2}\Big[g^{\mu\nu}(D_\mu^{(n)}\phi_n)^*(D_\nu^{(n)}\phi_n) - M_n^2\phi_n^*\phi_n - \mathscr{E}\Big(R_4 - \frac{1}{4}k^2F_{\mu\nu}F^{\mu\nu}\Big)\phi_n^*\phi_n\Big] \tag{3.8a}$$

where

$$D_\mu^{(n)}\phi_n = \nabla_\mu\phi_n + 2q_nA_\mu\phi_n \tag{3.8b}$$

$$M_n^2 = M_0^2 + \frac{(n+\alpha)^2}{b^2}M^2 - \frac{3}{2}\frac{\dot a}{a}\frac{\dot b}{b} - \frac{1}{4}\left(\frac{\dot b}{b}\right)^2 - \frac{1}{2}\frac{d}{dt}\left(\frac{\dot b}{b}\right) \tag{3.8c}$$

and

$$q_n = (n+\alpha)\,e = (n+\alpha)kM \tag{3.8d}$$

Here q_n is charge of the scalar field in the nth mode, which is an integral (half-integral) multiple of e ($=kM$) for untwisted (twisted) fields.

4. ONE-LOOP EFFECTIVE ACTION

The one-loop effective action for ϕ_n is calculated for the nth mode and summed up for all modes to get

$$\Gamma_\phi^{(1)} - \frac{i}{2}\sum_{n=-\infty}^{\infty} \ln\det\Delta_n \tag{4.1}$$

where Δ_n is the operator defined as

$$\Delta_n = g^{\mu\nu}D_\mu^{(n)}D_\nu^{(n)} + M_n^2 + \mathscr{E}(R_4 - \tfrac{1}{4}k^2F_{\mu\nu}F^{\mu\nu}) \tag{4.2}$$

Using the kernel $K_n(s, x, x)$ for Δ_n, we can rewrite (4.1) as

$$\Gamma_\phi^{(1)} = -\frac{i}{2}\sum_{n=-\infty}^{\infty}\int d^4x\,(-g_4)^{1/2}\int_0^\infty \frac{ds}{s}\,\mathrm{tr}\,K_n(s, x, x) \tag{4.3}$$

where

$$K_n(s, x, x) = i\mu^{4-N}(4\pi is)^{-N/2}\exp(-iM_{(n)}^2 s)\sum_{k=0}^{\infty}(is)^k a_k(x)$$

(N is the space-time dimension used as a dimensional regulator with $N \to 4$, and μ is a constant of mass dimension to get dimensionless action). For Δ_n given by (4.2) (Toms, 1983; Dewitt, 1965, 1975; Gilkey, 1973)

$$a_0(x) = 1 \tag{4.4a}$$

$$a_1(x) = (\tfrac{1}{6} - \mathscr{E})R_4 + \tfrac{1}{4}\mathscr{E}k^2 F_{\mu\nu}F^{\mu\nu} \tag{4.4b}$$

$$a_2(x) = -\tfrac{1}{12}k^2 M^2 (n+\alpha)^2 F^{\mu\nu}F_{\mu\nu} + \cdots \tag{4.4c}$$

(only relevant terms are mentioned here).

Integrating over s in (4.3) and using (4.4), we find

$$\begin{aligned}\Gamma_\phi^{(1)} = &-\frac{1}{2(4\pi)^2}\int d^4x\,(-g_4)^{1/2}\\ &\times\left\{\lim_{N\to 4}\sqrt{-\frac{N}{2}}\sum_{n=-\infty}^{\infty}\left[\frac{(n+\alpha)^2M^2}{b^2}+\tilde{M}^2(t)\right]^{N/2}\right.\\ &+\lim_{N\to 4}\sqrt{1-\frac{N}{2}}\sum_{n=-\infty}^{\infty}\left[\frac{(n+\alpha)^2M^2}{b^2}+\tilde{M}^2(t)\right]^{N/2-1}\left(\frac{1}{6}-\mathscr{E}\right)R_4\\ &+\lim_{N\to 4}\left\{\frac{1}{4}\mathscr{E}k^2\sqrt{1-\frac{N}{2}}\sum_{n=-\infty}^{\infty}\left[\frac{(n+\alpha)^2M^2}{b^2}+\tilde{M}^2(t)\right]^{N/2-1}\right.\\ &\left.\left.-\frac{1}{2}\sqrt{2-\frac{N}{2}}\sum_{n=-\infty}^{\infty}k^2M^2(n+\alpha)^2\left[\frac{(n+\alpha)^2M^2}{b^2}+\tilde{M}^2(t)\right]^{N/2-2}\right\}\right.\\ &\times F_{\mu\nu}F^{\mu\nu}+\cdots\end{aligned} \tag{4.5}$$

where

$$\tilde{M}^2(t) = -\frac{3}{2}\frac{\dot{a}}{a}\frac{\dot{b}}{b} - \frac{1}{4}\left(\frac{\dot{b}}{b}\right)^2 - \frac{1}{2}\frac{d}{dt}\left(\frac{\dot{b}}{b}\right)$$

Using formula (B6) of Ford (1989),

$$\sum_{n=-\infty}^{\infty}[(n+c)^2+d^2]^{-\lambda} = \pi^{1/2}d^{1-2\lambda}\sqrt{\frac{\lambda-1/2}{\lambda}} + 4\sin\pi\lambda f_\lambda(c, d) \tag{4.6}$$

(where $\operatorname{Re}\lambda > 1/2$ and c and d are real), we can sum the series in (4.5) for $\tilde{M}^2(t) > 0$ to yield, when $\alpha = 0$,

$$\begin{aligned}\Gamma_\phi^{(1)} = &-\frac{1}{2(4\pi)^2}\int d^4x\,(-g_4)^{1/2}\left[-\frac{8\pi}{15}\tilde{M}^5(t)\frac{b}{M}+\frac{4\pi b}{3M}\tilde{M}^3(t)+\left(\frac{1}{6}-\mathscr{E}\right)R_4\right.\\ &\left.+\frac{k^2}{4}\left(\frac{4\pi\mathscr{E}b}{3M}\tilde{M}^3(t)+\frac{M^2\zeta(3)}{24\pi^2}\right)F_{\mu\nu}F^{\mu\nu}+\cdots\right]\end{aligned} \tag{4.7}$$

[where $\zeta(p)$ is the Riemann zeta function] and when $\alpha = 1/2$

$$\Gamma_{\phi}^{(1)} = \frac{1}{2(4\pi)^2}\int d^4x\,(-g_4)^{1/2}\left[-\frac{8\pi b}{15M}\tilde{M}^5(t) + \frac{4\pi b}{3M}\tilde{M}^3(t)\left(\frac{1}{6} - \mathscr{E}\right)R_4 + \frac{k^2}{4}\left(\frac{4\pi\mathscr{E}b}{3M}\tilde{M}^3(t) - \frac{M^2\zeta(3)}{4\pi^2}\right)F_{\mu\nu}F^{\mu\nu} + \cdots\right] \tag{4.8}$$

If N_0^+ (N_0^-) is the number of untwisted (twisted) scalar fields, we have

$$\Gamma_{\phi}^{(1)} = -\frac{1}{2(4\pi)^2}\int d^4x\,(-g_4)^{1/2} \times\left\{-\frac{8\pi b}{15m}\tilde{M}^5(t)(N_0^+ + N_0^-) + \frac{4\pi b}{3M}(N_0^+ + N_0^-)\tilde{M}^3(t)\left(\frac{1}{6} - \mathscr{E}\right)R_4 + \frac{k^2}{4}\left[\frac{4\pi\mathscr{E}b}{3M}\tilde{M}^3(t)(N_0^+ + N_0^-) + \frac{M^2\zeta(3)}{6\pi^2}\left(N_0^+ - \frac{3}{2}N_0^-\right)\right] \times F_{\mu\nu}F^{\mu\nu} + \cdots\right\} \tag{4.9}$$

From (3.6) and (4.9) we can write the effective action as

$$S_{\rm eff}^{(4)} = \int d^4x(-g_4)^{1/2}\left\{\frac{b}{60\pi M}\tilde{M}^5(t)(N_0^+ + N_0^-) - \left[\frac{b}{24\pi M}(N_0^+ + N_0^-)\tilde{M}^3(t)\left(\frac{1}{6} - \mathscr{E}\right) + \frac{b}{16\pi\bar{G}}\right]R_4 + \frac{k^2}{4}\left[\frac{b}{16\pi\bar{G}} + \frac{4\pi\mathscr{E}b}{3M}\tilde{M}^3(t)(N_0^+ + N_0^-) + \frac{M^2\zeta(3)}{6\pi^2}\left(N_0^+ - \frac{3}{2}N_0^-\right)\right] \times F_{\mu\nu}F^{\mu\nu} + \cdots\right\} \tag{4.10}$$

According to (4.10), the effective gravitational constant and cosmological constant are given by

$$\frac{1}{16\pi G_{\rm eff}} = \frac{b}{16\pi\bar{G}} + \frac{b}{24\pi M}(N_0^+ + N_0^-)\tilde{M}^3(t)\left(\frac{1}{6} - \mathscr{E}\right) \tag{4.11}$$

and

$$\frac{\Lambda_{\rm eff}}{8\pi G_{\rm eff}} = \frac{b}{60\pi M}\tilde{M}^5(t)(N_0^+ + N_0^-) \tag{4.12}$$

Thus, one finds that $G_{\rm eff}$ and $\Lambda_{\rm eff}$ are time dependent, but to have a clear picture one needs the function $b(t)$ explicitly. Here $b(t)$ is the solution of five-dimensional Einstein equations. Moreover, using the normalization condition for A_μ, one gets from (4.10)

$$\frac{b(t)}{16\pi\bar{G}} + \frac{4\pi\mathscr{E}b(t)}{3M}\tilde{M}^3(t)(N_0^+ + N_0^-) + \frac{M^2\zeta(3)}{6\pi^2}\left(N_0^+ - \frac{3}{2}N_0^-\right) = \frac{M^2}{e^2} \tag{4.13}$$

It is interesting to see from (4.13) that e (the gauge coupling constant for the electromagnetic field) is time dependent. Also one gets the time dependence of the fine structure constant as

$$\frac{e^2}{4\pi} = \frac{M^2}{4\pi}\left[\frac{b(t)}{16\pi\bar{G}} + \frac{4\pi\mathscr{E}b(t)}{3M}\tilde{M}^3(t)(N_0^+ + N_0^-) + \frac{M^2\zeta(3)}{6\pi^2}\left(N_0^+ - \frac{3}{2}N_0^-\right)\right]^{-1} \tag{4.14}$$

It is clear from (4.11) and (4.12) that if $\lim_{t\to\infty} b(t) = 0$, $G_{\rm eff}\to\infty$ as $t\to\infty$, and also $\Lambda_{\rm eff}$ becomes indeterminate. To avoid these kinds of unphysical situations, one needs $b(t)\to f$ (const) > 0 as $t\to\infty$.

5. COSMOLOGICAL SOLUTION OF EINSTEIN'S FIELD EQUATIONS

On varying the action (2.3) with respect to the metric tensor, one gets the Einstein field equations

$$G_{m'n'} = R_{m'n'} - \frac{1}{2}g_{m'n'}R = 8\pi G\left[T^{(\phi)}_{m'n'} + \frac{k^2}{16\pi G}T^{(F)}_{m'n'} + T^{(m)}_{m'n'}\right] \tag{5.1}$$

where $G_{m'n'}$ is the Einstein tensor, $T^{(\phi)}_{m'n'}$ is the energy momentum tensor for ϕ given as

$$\begin{aligned} T^{(\phi)}_{m'n'} = (D_{m'}\phi)^*(D_{n'}\phi) - \frac{1}{2}g_{m'n'}[g^{r's'}(D_{r'}\phi)^*(D_{s'}\phi) \\ - \mathscr{E}R_5\phi^*\phi - M_0^2\phi^*\phi] \end{aligned} \tag{5.2}$$

$T^{(m)}_{m'n'}$ is the energy-momentum tensor for matter (perfect fluid), and $T^{(F)}_{m'n'}$ is the energy-momentum tensor for the gauge field A given as

$$T^{(F)}_{m'n'} = F_{m'l'}F^{l'}_{n'} - \frac{1}{4}g_{m'n'}F_{r's'}F^{r's'} \tag{5.3a}$$

Requiring that A_μ lives on M^4 only, due to the homogeneity of M^3 (three-dimensional subspace of M^4), one can write Maxwell's fields $F_{m'n'}$ as

$$F^{m'n'} = \begin{cases} \dfrac{\epsilon^{m'n'}}{(-g_4)^{1/2}}F(t) & \text{for} \quad m' = \mu, \quad n' = \nu \\ 0 & \text{otherwise} \end{cases} \tag{5.3b}$$

where $\epsilon^{m'n'}$ is the Levi–Civita tensor. Using (5.3b) in (5.3a), we obtain

$$T^{(F)}_{m'n'} = \bar{\epsilon} g_{m'n'} F^2(t) \tag{5.3c}$$

where

$$\bar{\epsilon} = \begin{cases} -1 & \text{for} \quad m' = \mu, \quad n' = \nu \\ 0 & \text{otherwise} \end{cases}$$

Maxwell's fields $F_{m'n'}$ satisfy the equation

$$\frac{1}{(-g_5)^{1/2}} \partial_{m'}((-g_5)^{1/2} F^{m'n'}) = 0$$

which yields

$$F(t) = \frac{A}{b(t)} \tag{5.4}$$

using (5.3b).

Now, with the total energy density $\rho = \rho^{(\phi)} + \rho^{(m)}$ the pressure on the three-dimensional spatial subspace of the external manifold $p = p^{(\phi)} + p^{(m)}$, and the pressure on the internal manifold $p = p^{(\phi)} + p^{(m)}$, the energy-momentum tensor is written as (Gleiser and Taylor, 1983)

$$T^{(\phi,m)}_{m'n'} = (\rho + p) u_{m'} u_{n'} - (\delta p + \delta' P) g_{m'n'} \tag{5.5}$$

where $u^{m'} = (1, 0, 0, 0, 0)$ is the velocity vector normalized to unity,

$$\delta = \begin{cases} 1 & \text{for} \quad m', n' = 0, 1, 2, 3 \\ 0 & \text{for} \quad m', n' = 4 \end{cases} \tag{5.6a}$$

$$\delta' = \begin{cases} 0 & \text{for} \quad m', n' = 0, 1, 2, 3 \\ 1 & \text{for} \quad m', n' = 4 \end{cases} \tag{5.6b}$$

and indices ϕ, m over the variables correspond to scalar field and matter, respectively. Due to the spatial homogeneity of the model, ρ, p, and P are only time dependent. From (5.5) one gets

$$T^0_0 = \rho, \qquad T^i_i = p \qquad (i = 1, 2, 3) \tag{5.7}$$

and

$$T^4_4 = \mathrm{P}$$

The conservation of total energy-momentum tensor

$$T^{(\phi,m,F)}_{m'n'} = T^{(\phi,m)}_{m'n'} + T^{(F)}_{m'n'}$$

yields

$$\dot{\rho} + \rho\left(\frac{3\dot{a}}{a} + \frac{\dot{b}}{b}\right) + 3p\,\frac{\dot{a}}{a} + \rho\,\frac{\dot{b}}{b} - \frac{\partial}{\partial t}[F^2(t)] = 0 \tag{5.8}$$

Now the Einstein field equations are written as

$$G_0^0 = \frac{6k_3 t_p^2}{a^2 l^2} + \left(\frac{3a'}{a} + \frac{b'}{b}\right)^2 - 3\left(\frac{a'}{a}\right)^2 - \left(\frac{b'}{b}\right)^2$$

$$= 8\pi G t_p^2[\rho^{(m)} + \rho^{(\phi)} - F^2(t)] \tag{5.9a}$$

$$\frac{2k_3 t_p^2}{a^2 l^2} + \frac{d}{d\tilde{t}}\left(\frac{a'}{a}\right) + \frac{a'}{a}\left(\frac{3a'}{a} + \frac{b'}{b}\right) = 8\pi G t_{\mathrm{P}}^2\left(p - \frac{F^2}{3}\right) \tag{5.9b}$$

$$\frac{d}{d\tilde{t}}\left(\frac{b'}{b}\right) + \frac{b'}{b}\left(\frac{3a'}{a} + \frac{b'}{b}\right) = 8\pi G t_{\mathrm{P}}^2\left(P - \frac{4F^2}{3}\right) \tag{5.9c}$$

where $\tilde{t} = t/t_{\mathrm{P}}$ and prime denotes $\partial/\partial\tilde{t}$.

At the end of Section 4, the required nature for the solution $b(t)$ has been discussed. To have such a solution, it is assumed that

$$b^2 = f^2 + \frac{1}{a^2(t)} \tag{5.10}$$

Equation (5.10) satisfies the criterion $\lim_{t\to\infty} b(t) = f$.

Using (5.10) and (5.9b) and (5.9c), one gets

$$\frac{2k_3 t_{\mathrm{P}}^2}{a^2 l^2} + \frac{2f^2 a'^2}{1 + f^2 a^2} = 8\pi G t_{\mathrm{P}}^2\left[P + (1 + f^2 a^2)P - \frac{5 + 4f^2 a^2}{3} F^2\right] \tag{5.11}$$

In order to solve (5.11), one needs an equation relating p, P, and $F^2(t)$. But no such relation is available. Nevertheless, one may impose the condition

$$p^{(\phi)} + p^{(m)} + (1 + f^2 a^2)[P^{(\phi)} + P^{(m)}] = \frac{5 + 4f^2 a^2}{3} F^2 \tag{5.12}$$

In (5.12), $p^{(\phi)}$ and $P^{(\phi)}$ can be calculated from (5.2) and (5.3), and $F^2(t)$ is given by (5.4). But $p^{(m)}$ and $P^{(m)}$ are still arbitrary. Condition (5.12) may be accepted provided that it is satisfactory from the physical point of view. From (5.12), obviously $p^{(m)}$ and $P^{(m)}$ have time dependence only (due to the spatial homogeneity of the model) and $p^{(\phi)}$, $P^{(\phi)}$, $F(t)$, and $a(t)$ have time dependence only.

Utilizing condition (5.12), we can exactly integrate (5.11) to

$$1 + f^2 a^2 = \left(\frac{ft}{l} + \alpha\right)^2 \tag{5.13}$$

provided that $k_3 = -1$. If $k_3 = +1$, (5.12) yields a complex solution. The solution is constant if $k_3 = 0$. So only $k_3 = -1$ is considered hereafter.

Solutions (5.10) and (5.13) can be accepted if they satisfy (5.9a). From the conservation equations one gets

$$\frac{\partial}{\partial t} G_0^0 = 0 = \frac{\partial}{\partial t} T_0^0 \tag{5.14}$$

in the model considered here. Equation (5.14) implies that if the constraint equation (5.9a) is satisfied at one particular epoch, it is satisfied by the above solutions at every epoch. So, one can choose a particular epoch $t = 0$.

We integrated (5.8) to

$$\rho = \rho_0 - \frac{1}{a^3 b} \int_0^t \left(3p \frac{\dot{a}}{a} + P \frac{\dot{b}}{b} - \frac{\partial}{\partial t} F^2 \right) a^3 b \, dt' \tag{5.15}$$

Now (5.9a) is satisfied at $t = 0$ by solutions (5.10) and (5.13) provided

$$\frac{-6t_P^2 f^2(\tilde{\alpha}^2 - 1)}{l^2} + \frac{f^2(3\tilde{\alpha}^2 + 1)^2 t_P^2}{l^2 \alpha^2} - \frac{3\tilde{\alpha}^2 f^2 t_P^2}{l^2} - \frac{f^2(\tilde{\alpha}^2 - 1) t_P^2}{\alpha^2 l^2}$$
$$= 8\pi G t_P^2 \left[\rho_0 (\tilde{\alpha}^2 - 1)^2 - \frac{A^2(\tilde{\alpha}^2 - 1)^3}{f^2 \alpha^2} \right]$$

which yields

$$\alpha^2 - 1 \simeq \frac{7f^2 t_P^2 / l^2 + [(49 f^4 / l^4) t_P^4 + (52 f^2 t_P^2 / l^2)(8\pi G t_P^2 \rho_0 - 6 f^2 t_P^2 / l^2)]^{1/2}}{2(8\pi G t_P^2 \rho_0 - 6 f^2 t_P^2 / l^2)} \tag{5.16}$$

ρ_0 is the energy density at $t = 0$, and is expected to be quite large. So the value of α^2 given by (5.16) is reasonable. Thus it is found that $a(t)$ given by the solution (5.13) is nonsingular, provided that ρ_0 is finite.

6. CONCLUDING REMARKS

In this section, results given by equations (4.11), (4.12), and (4.14) are interpreted in the light of solutions of the five-dimensional Einstein equations obtained in the preceding section. These solutions are

$$a(t) = f^{-1} \left[\left(\frac{ft}{l} + \tilde{\alpha} \right)^2 - 1 \right]^{1/2} \tag{6.1}$$

and

$$b(t) = f \left(\frac{ft}{l} + \tilde{\alpha} \right) \left[\left(\frac{ft}{l} + \tilde{\alpha} \right)^2 - 1 \right]^{-1/2} \tag{6.2}$$

Now, equations (4.11), (4.12), and (4.14) are rewritten as

$$\frac{1}{16\pi G_{\text{eff}}} = \frac{f(ft/l+\tilde{\alpha})}{16\pi\bar{G}[(ft/l+\tilde{\alpha})^2-1]^{1/2}} - \frac{5(N_0^+ + N_0^-)\tilde{M}^3(t)f(ft/l+\tilde{\alpha})}{576\pi M[ft/l+\tilde{\alpha})^2-1]^{1/2}} \tag{6.3}$$

$$\frac{\Lambda_{\text{eff}}}{8\pi G_{\text{eff}}} = \frac{f(ft/l+\tilde{\alpha})\tilde{M}^5(t)(N_0^+ + N_0^-)}{60\pi M[(ft/l+\tilde{\alpha})^2-1]^{1/2}} \tag{6.4}$$

and

$$\frac{e^2}{4\pi} = \frac{M^2}{4\pi}\left[\frac{f(ft/l+\tilde{\alpha})}{16\pi\bar{G}[(ft/l+\tilde{\alpha})^2-1]^{1/2}} + \frac{4\pi\mathscr{E}f(ft/l+\tilde{\alpha})}{3M[(ft/l+\tilde{\alpha})^2-1]^{1/2}}\right.$$
$$\left.\times \tilde{M}^3(t)(N_0^+ + N_0^-) + \frac{M^2\rho(3)}{6\pi^2}\left(N_0^+ - \frac{3}{2}N_0^-\right)\right]^{-1} \tag{6.5}$$

$\tilde{M}(t)$ in (6.3)–(6.5) is given by

$$\tilde{M}^2(t) = \frac{6(1-f^2)(ft/l+\alpha)^2 + 2f^2 - 1}{4f^2l^2(ft/l+\alpha)^2[(ft/l+\alpha)^2-1]^2} \tag{6.6}$$

As $t \to \infty$, from equation (6.3) one gets

$$G_{\text{eff}} = \bar{G}/f \tag{6.7}$$

The present value of the gravitational constant is G_{N} (Newtonian gravitational constant) so $\lim_{t\to\infty} G_{\text{eff}} = G_{\text{N}}$; hence,

$$\bar{G} = fG_{\text{N}} \tag{6.8}$$

It is also known that the fine structure constant at low energy is 1/137. Hence one gets from equations (6.5) and (6.8)

$$137 \simeq \frac{4\pi}{M^2}\left[\frac{1}{16\pi G_{\text{N}}} + \frac{M^2\zeta(3)}{6\pi^2}\left(N_0^+ - \frac{3}{2}N_0^-\right)\right]$$

which provides a compactification mass scale as

$$M = \frac{M_{\text{P}}}{2}\left[137 - \frac{2\zeta(3)}{3\pi}\left(N_0^+ - \frac{3}{2}N_0^-\right)\right]^{-1/2} \tag{6.9}$$

Equation (6.4) implies that

$$\lim_{t\to\infty} \Lambda_{\text{eff}} = 0 \tag{6.10}$$

Also, for large t, the effective radius of the circle $b(t)\bar{\rho}$ ($\bar{\rho}$ is the physical radius of the circle) is approximately equal to $f\bar{\rho}$. So

$$f\bar{\rho} \lesssim L_{\text{P}} \qquad \text{(Planck length)} \tag{6.11}$$

because the extra dimension is not observed.

Now, from equations (6.9) and (6.11)

$$f \lesssim \frac{1}{2[137 - [2\zeta(3)/3\pi](N_0^+ - \frac{3}{2}N_0^-)]^{1/2}} \tag{6.12}$$

It is noted from (6.12) that the entire analysis is valid so long as $N_0^+ - \frac{3}{2}N_0^- \leq 536$.

To get a better feeling for the results we discuss the special case for which $N_0^+ - \frac{3}{2}N_0^- = 536$. In this case, one gets from (6.12) that

$$f \simeq 1 \qquad \text{and} \qquad M \simeq M_P \tag{6.13}$$

Using (6.13) and (6.6), we obtain

$$\tilde{M}(t) \simeq \frac{1}{2l(ft/l + \tilde{\alpha})[(ft/l + \tilde{\alpha})^2 - 1]} \tag{6.14}$$

Connecting (6.14) and (6.3), we find

$$\begin{aligned} G_{\text{eff}} = {} & 36M_P l^3 \left[\left(\frac{ft}{l} + \tilde{\alpha}\right)^2 - 1\right]^{3/2} \\ & \times \left\{36M_P^3 l^3 \left(\frac{ft}{l} + \tilde{\alpha}\right)\left[\left(\frac{ft}{l} + \tilde{\alpha}\right)^2 - 1\right] - 5(N_0^+ + N_0^-)\right\}^{-1} \end{aligned} \tag{6.15}$$

$$\begin{aligned} \Lambda_{\text{eff}} = {} & 3(N_0^+ + N_0^-)\left\{l^2\left(\frac{ft}{l} + \tilde{\alpha}\right)^4\left[\left(\frac{ft}{l} + \tilde{\alpha}\right)^2 - 1\right]^4 \right. \\ & \left. \times \left[36M_P^3 l^3 \left(\frac{ft}{l} + \tilde{\alpha}\right)\left(\frac{ft}{l} + \tilde{\alpha}\right)^2 - 1\right] - 5(N_0^+ + N_0^-)\right]\right\}^{-1} \end{aligned} \tag{6.16}$$

and

$$\begin{aligned} \frac{e^2}{4\pi} = {} & \left\{136.72 + \frac{(ft/l + \tilde{\alpha})}{4[(ft/l + \tilde{\alpha})^2 - 1]^{1/2}} \right. \\ & \left. + \frac{6\pi^2(N_0^+ + N_0^-)}{3M_P^3 l^3 (ft/l + \tilde{\alpha})^2[(ft/l + \tilde{\alpha})^2 - 1]^{7/2}}\right\}^{-1} \end{aligned} \tag{6.17}$$

The quantity ρ_0, which is the density of the universe at $t = 0$, is supposed to be extremely large, but it is not exactly known. So, for a computer plot of the results, ρ_0 is assumed to be $\sim 10^{82}$ GeV^4. Using this assumption, from (5.16) we obtain

$$\tilde{\alpha}^2 - 1 \simeq 5.09 \times 10^{-23} \tag{6.18}$$

Also we assume that $N_0^+ = 536$ and $N_0^- = 0$. In this case, as obtained above, $f \simeq 1$. With these assumptions we plot graphs for $a(t)$, $b(t)$, G_{eff}, Λ_{eff}, and $e^2/4\pi$. Figure 1 is a graph of $a(\tilde{t})$ versus $\tilde{t}$, which shows a sharp increase

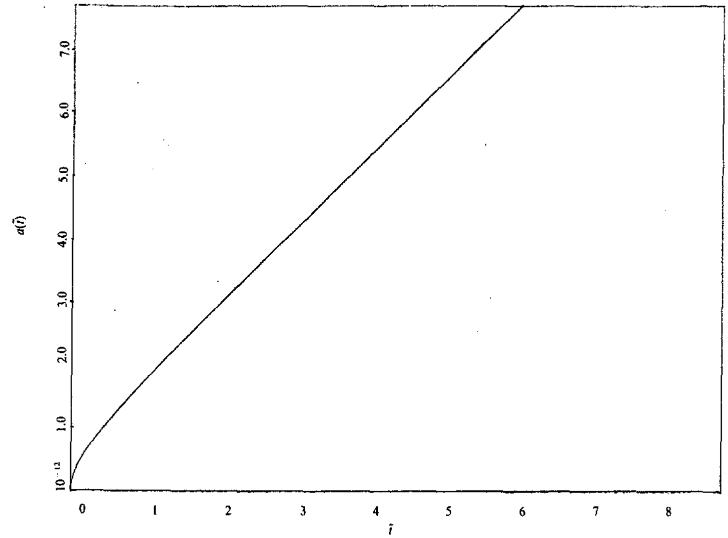

Fig. 1

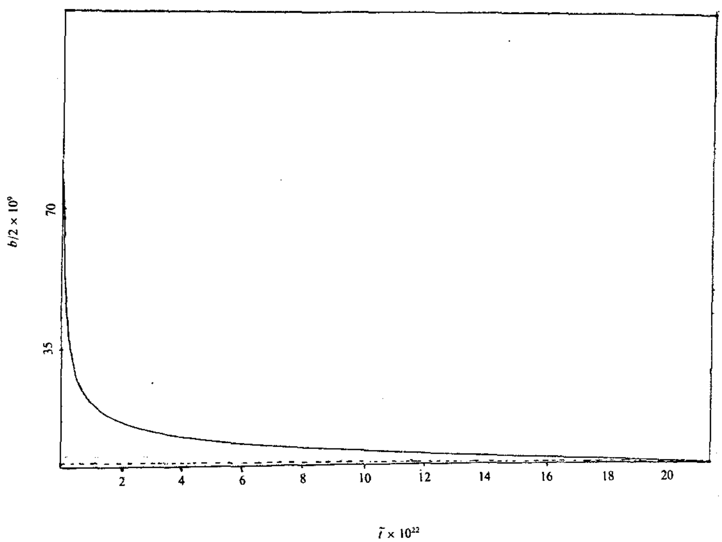

Fig. 2

of $a(\tilde{t})$. Figure 2 is a graph of $b(\tilde{t})/(2 \times 10^9)$ versus $\tilde{t} \times 10^{22}$, which shows that in the beginning $b(t)$ is very large followed by a sharp fall, with stability of $b(t)$ around 1 near $t = 56.8 t_P$ (t_P is the Planck time). In Fig. 3, G_{eff} is plotted against $\tilde{t}$, showing that at $t = 0$, G_{eff} is zero. It sharply

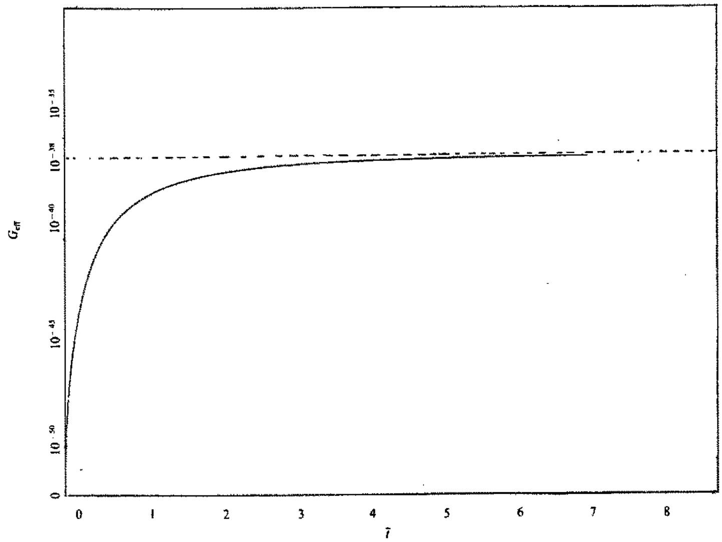

Fig. 3

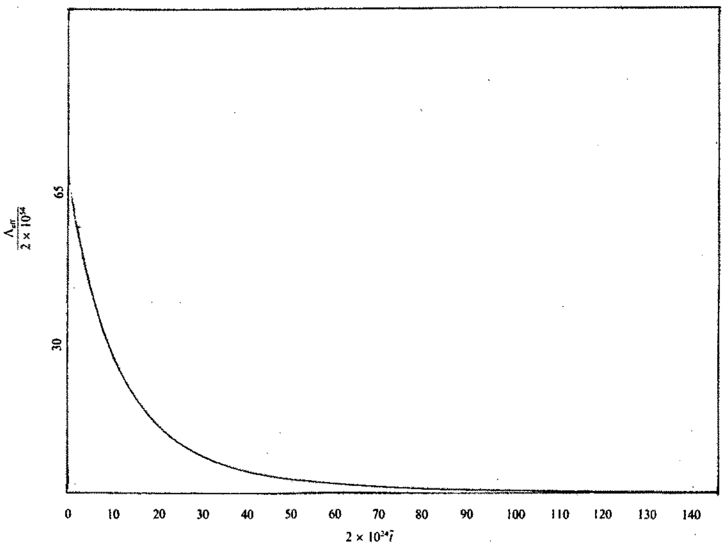

Fig. 4

increases and is stabilized around G_{N} (which is equal to $10^{-38}\,\text{GeV}^{-2}$). Figure 4 is a plot of $\Lambda_{\text{eff}}/(2\times10^{54})$ against $2\times10^{24}\tilde{t}$. This graph shows that at $t=0$, $\Lambda_{\text{eff}}\simeq10^{56}\,\text{GeV}^2$ and around $\tilde{t}=7.4\times10^{-23}$, $\Lambda_{\text{eff}}\simeq0$. In Fig. 5, $(e^2/4\pi)\times10^4$ is plotted against $\tilde{t}\times10^4$, which shows that in a very short

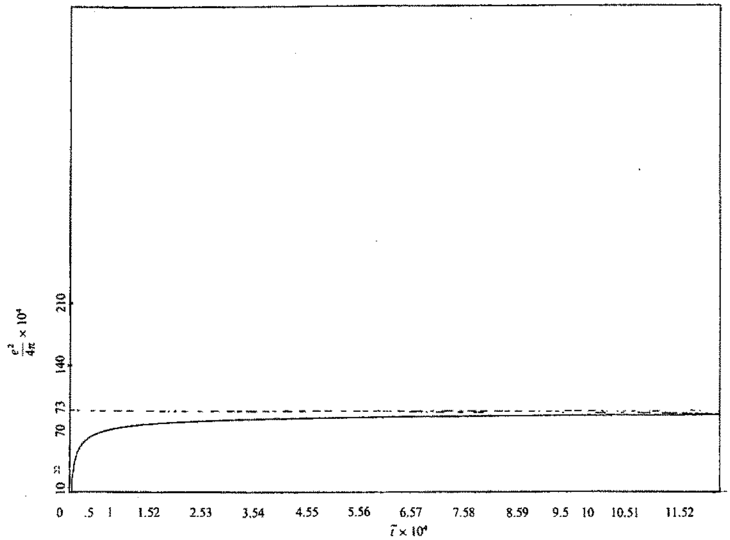

Fig. 5

time $1.15 \times 10^{-3} t_P$, the fine structure constant approaches the observed value 1/137. Thus, according to the discussion given above, one finds that within the Planck time, the fundamental constants acquire the observed values.

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